Strong universality in forced and decaying turbulence in a shell model

Victor S. L'vov,[1] Rubén A. Pasmanter,[1,2] Anna Pomyalov,[1] and Itamar Procaccia[1,3]
[1]*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*Royal Dutch Meteorological Institute, P.O. Box 201, 3730 AE De Bilt, The Netherlands*
[3]*Department of Physics, the Chinese University of Hong Kong, Shatin, Hong Kong*
(Received 18 February 2003; published 27 June 2003)

The weak version of universality in turbulence refers to the independence of the scaling exponents of the nth order structure functions from the statistics of the forcing. The strong version includes universality of the coefficients of the structure functions in the isotropic sector, once normalized by the mean energy flux. We demonstrate that shell models of turbulence exhibit strong universality for both forced and decaying turbulence. The exponents and the normalized coefficients are time independent in decaying turbulence, forcing independent in forced turbulence, and equal for decaying and forced turbulence. We conjecture that this is also the case for Navier-Stokes turbulence.

DOI: 10.1103/PhysRevE.67.066310 PACS number(s): 47.27.Gs, 47.10.+g

I. INTRODUCTION

The statistical theory of fluid turbulence is concerned with correlation functions of the turbulent velocity vector field $\boldsymbol{u}(\boldsymbol{r},t)$, where $\boldsymbol{r}$ is the spatial position and t is the time [1]. Since the velocity field is a vector, multipoint and multitime correlation functions are, in general, tensor functions of the vector positions and the scalar times. Naturally such functions have rather complicated forms, which are difficult to measure and to compute. Consequently, almost from its very beginning, the statistical theory of turbulence had been discussed in the context of an isotropic and homogeneous model. The notion of isotropic turbulence was first introduced by Taylor in 1935 [2]. It refers to a turbulent flow, in which the statistical averages of every function of the velocity field and its derivatives with respect to a particular frame of axes is invariant to any rotation in the axes. This is a very effective mathematical simplification, which, if properly used, can drastically reduce the mathematical complexity of the theory. For this reason, it was very soon adopted by others, such as Kármán and Howarth [3] who derived the Kármán-Howarth equation, and Kolmogorov [4,5] who derived the 4/5 law. In fact, most of the theoretical work in turbulence in the past 60 years had been limited to the isotropic model.

Within the homogeneous and isotropic model there developed the notion of universality of turbulence. By universality, we mean the tendency of different turbulent systems to show, for very large Reynolds numbers Re, the same small-scale statistical behavior when the measurements are done far away from the boundaries. The statistical objects (see below for definitions) exhibit approximately the same scaling exponents whether they are measured in the atmospheric boundary layer, in a wind tunnel or in a computer simulation, provided they are measured far from the boundaries. Moreover, the accumulated experimental knowledge over the years indicated that not only in forced, stationary turbulence, but also in decaying turbulence, there is a regime of time where the statistical objects exhibit the same scaling properties. This phenomenon was explained [1] by the widely separated time scales (''eddy turn over times'') that characterize large and small length scales in turbulence. While turbulence was decaying on the time scale of the large eddies, the small one had ample time to reach an ''energy-flux equilibrium'' that in terms of scaling behavior was indistinguishable from forced turbulence. Thus, there exists a wide-spread belief that at least from the point of view of scaling exponents, forced and decaying turbulence are in the same universality class, sharing the same scaling exponents of the corresponding correlation functions.

To actually *prove* this type of universality in experiments and simulations is, however, far from straightforward. To achieve reasonable precision in the measurement of scaling exponents one needs large ranges of scales where scaling prevails, and this entails large Reynolds numbers. Unfortunately large Reynolds numbers are available usually when anisotropic effects are large, like in the atmospheric boundary layer or in large wind tunnels. Direct Numerical Simulations (DNS) can be used to eliminate anisotropy almost completely (up to lattice anisotropy that are unavoidable in simulations), but they are limited to relatively low Re, notwithstanding the very recent simulations of size 4096^3 [6]. Decaying turbulence is even harder to characterize precisely, since the effective Reynolds number decreases in time. Thus, actual measurements of scaling properties are fraught with difficulties, corrections to scaling, effects of anisotropy, and what not. As a result, over the years [7] and also very recently, it was proposed [8] that structure functions in forced and decaying turbulence have different exponents. In this paper we take a strong stand, proposing that the universality that actually exists in turbulence is even stronger than what has been anticipated so far.

To make our point clear, recall that the statistical description of fully developed turbulence employs correlation functions and structure functions. These are ensemble average of velocity differences across a length scale R. In the theoretical studies of turbulence the two most common ensemble averages are *over realizations of the forcing* when one studies forced turbulence, or *over initial conditions* when one studies decaying turbulence. The longitudinal structure functions are the simplest such objects, being moments of the longitudinal components of the velocity difference between two points.

We will denote the longitudinal structure functions in forced and decaying turbulence by S_p and F_p, respectively, with the precise definitions

$$S_p(R) \equiv \left\langle \left\{ [\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{R},t)-\boldsymbol{u}(\boldsymbol{r},t)] \cdot \frac{\boldsymbol{R}}{R} \right\}^p \right\rangle_f, \tag{1}$$

$$F_p(R,t) \equiv \left\langle \left\{ [\boldsymbol{u}(\boldsymbol{r}+\boldsymbol{R},t)-\boldsymbol{u}(\boldsymbol{r},t)] \cdot \frac{\boldsymbol{R}}{R} \right\}^p \right\rangle_i. \tag{2}$$

Here $\boldsymbol{u}(\boldsymbol{r},t)$ is the velocity field measured at point $\boldsymbol{r}$ at time t. $\langle \cdots \rangle_f$ and $\langle \cdots \rangle_i$ stand for ensemble averaging over the forcing and the initial conditions, respectively. In writing Eq. (1) we assumed that the forcing is stationary in time, homogeneous, and isotropic, and thus S_n is a function of the scalar R only. In writing Eq. (2) we assumed that the initial condition are homogeneous and isotropic. Of course, the decaying structure functions are by definition time dependent. The widely spread belief [1,9,10] is that, for R values in the inertial range of turbulence (much smaller than the forcing scale but much larger than the dissipation scale), the scaling exponents ζ_p that characterize $S_p(R)$, i.e., $S_p(\lambda R) = \lambda^{\zeta_p} S_p(R)$, are the same as the scaling exponents that characterize $F_p(R,t)$ for a given value of t. Of course, also here R should be well in the (time dependent) inertial range and t should be neither too small nor too large [11]. In the sequel we refer to the identity of only the scaling exponents of these two sets of objects (if it exists) as "the weak version of universality." As mentioned above the existence of the weak version of universality is by no means accepted by everybody in the field of turbulence. Since there is no *proof* of this universality, doubts of its existence linger, and, for example, in Ref. [8] it was concluded that the scaling exponents of the two families of statistical objects are *not* the same. We note, however, that in the same paper it was stated that the scaling exponents of the longitudinal and transverse structure functions are also not the same. It was shown recently, however, that such statements stem from incomplete treatments of the effects of anisotropy [12,13], leaving hope that the weak version of universality is still correct.

In fact, in this paper we will propose that not only the weak version of universality is correct, but in fact also a "strong version of universality" is applicable. By the latter we mean that once properly normalized, the structure functions F_p and S_p agree not only in exponents but also in amplitudes. In the context of the second-order structure function, this is not a new statement. The universality of ζ_2 and C_2 was already stated in the 80s by Yaglom [14] and Kader [15]. Analyzing hundreds of experiments made in different flows under different conditions, Sreenivasan in 1995 came to the conclusion that "the Kolmogorov constant C_2 is *more or less* universal, essentially independent of the flow as well as the Reynolds number (for $R_\lambda > 50$ or so), . . . with the average value of $C_2 \approx 0.53$ with a standard deviation of about 0.055" [16]. Nevertheless, the universality of C_2 and C_p for $p \leqslant 4$ is still under debate. For example, very recently in Ref. [17] the authors argued on the basis of a 256^3 DNS "in favor of an 'exponents only' universality scenario for forced turbulence."

We believe that this strong version of universality was never stated before, and the common thinking is that amplitudes depend in a nonuniversal way on details of the forcing or the preparation of the decaying turbulence. While true, we will argue that the freedom afforded by such details is very limited, amounting at the end to just *one* free number, which, once taken into account, the strong version of universality applies.

Besides being an issue of fundamental importance to turbulence, there is another reason for returning at this time to the correspondence between force and decaying turbulence. The reason is that the riddle of anomalous scaling of correlation and structure functions in forced turbulent advection (passive and active) has been solved recently. First, in the context of the nongeneric Kraichnan model of passive scalar advection [18], and then, in steps, for passive vectors [19,20], generic passive scalars and vectors [21–23], and finally for generic active scalars and vectors [24–26]. The common thread of this advance is that anomalous scaling is discussed in the context of the decaying (unforced problem), in which one shows that there exist statistically preserved structures (eigenfunctions of eigenvalue 1 of the appropriate propagator of the decaying correlation functions). The decaying problem is independent of forcing, and one shows that the statistics of the forced problem is dominated by the same statistically preserved structures that are identified in the decaying problem. The calculation of the anomalous exponents boils down then to calculating eigenfunctions of linear operators. In these problems the correspondence between the decaying and forced statistics is proven mathematically or demonstrated beyond reasonable doubt by careful numerics. A crucial ingredient in all this progress is that turbulent advection is described by *linear* partial differential equations. There is, therefore, an urgent question how to translate (if it is possible) the newly acquired insights to the nonlinear turbulent problem itself, be it the Navier-Stokes equations or any of the shell models that were frequently discussed recently in the context of anomalous scaling. In this paper we make a step in this direction, analyzing the decay of the Sabra shell model [27] and showing numerically that the statistics of the decaying state and the forced turbulent state *are the same* in exponents *and in amplitudes* up to one freedom (the time-dependent mean energy). We opt to work with the shell model rather than the Navier-Stokes equations simply because the accuracy required for our aims exceeds the available scaling ranges and decay times for the latter. We express a strong belief that very similar results can be demonstrated also for Navier-Stokes turbulence. Indeed, in a future publication we will present the theory that stands behind the present numerical findings and demonstrate that the basic structure of that theory is the same for shell models and the Navier-Stokes equations.

The paper is organized as follows. In Sec. II we introduce the shell model and the numerical simulations that we perform. We present the data for the energy decay and explain what is the time domain for which we should compare the

TABLE I. Universal coefficients C_p and scaling exponents ζ_p obtained from the best fit of the numerical data. The auxiliary fit parameters in Eq. (17) are found to be in the intervals $\alpha_p \sim 0.5$–2 and $\mu_p \sim 0.6$–2.3. The error bars for each parameter correspond to the error function $\mathcal{E}$, Eq. (18), equal $\simeq \sqrt{2}\mathcal{E}_{\min}$ with all other parameters set to their optimal values.

	Time	C_2	ζ_2	$n_{d,2}$	C_4	ζ_4	$n_{d,4}$	C_6	ζ_6	$n_{d,6}$
Forced 1		0.73 ± 0.07	0.728 ± 0.006	17.0	0.60 ± 0.08	1.254 ± 0.008	16.24	0.62 ± 0.15	1.72 ± 0.01	15.87
Forced 2		0.73 ± 0.07	0.728 ± 0.006	17.0	0.60 ± 0.08	1.254 ± 0.008	16.24	0.61 ± 0.15	1.72 ± 0.01	15.87
Decay 1	20	0.72 ± 0.05	0.728 ± 0.006	17.2	0.62 ± 0.06	1.254 ± 0.008	17.0	0.79 ± 0.15	1.72 ± 0.01	16.4
	100	0.72 ± 0.06	0.728 ± 0.006	16.2	0.62 ± 0.07	1.254 ± 0.008	16.0	0.79 ± 0.15	1.72 ± 0.01	15.0
	10^3	0.72 ± 0.07	0.728 ± 0.006	13.5	0.61 ± 0.08	1.255 ± 0.008	13.2	0.77 ± 0.15	1.72 ± 0.01	13.0
	10^4	0.72 ± 0.08	0.728 ± 0.006	11.0	0.61 ± 0.09	1.256 ± 0.008	10.2	0.75 ± 0.15	1.73 ± 0.01	9.8
Decay 2	100	0.73 ± 0.1	0.73 ± 0.01	15.5	0.6 ± 0.2	1.25 ± 0.01	15.2	0.70 ± 0.20	1.72 ± 0.02	14.3
	10^3	0.74 ± 0.1	0.73 ± 0.01	13.3	0.6 ± 0.15	1.26 ± 0.01	13.1	0.66 ± 0.20	1.72 ± 0.02	12.4
	10^4	0.72 ± 0.1	0.73 ± 0.01	10.7	0.6 ± 0.2	1.25 ± 0.01	10.5	0.72 ± 0.25	1.73 ± 0.03	9.8

decaying and the forced statistics. In Sec. III we present the results for forced structure functions for different types of forcing. In determining the exponents *and the amplitudes* of these functions one has to be extra careful—we explain that one needs to find fits to functions throughout their range of existence. It is not enough to plot log-log plots for the inertial range. In Sec. IV we present the data for the decaying correlation functions, and explain how to find their exponents and amplitudes once the time dependence is taken into account. We explain theoretically that the decaying structure functions contain subleading contributions that decay fast toward small scales and do not affect the leading scaling exponents. The results of our calculations are summarized in Table I, which is the central result of this paper, giving strong support to the conjecture of *strong* universality. In Sec. V we present a summary and some concluding remarks.

II. MODEL AND ENERGY DECAY

A. Model and objectives

The Sabra shell model [27], like all shell models, is a reduced dynamical description of turbulence in terms of complex variables u_n, which represent velocity amplitudes associated with wave number $k_n = k_0\lambda^n$. The equations of motion are

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}u^*_{n+1} + bk_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \tag{3}$$

where the asterisk stands for complex conjugation and ν is the "viscosity." The coefficients a, b, and c are chosen such that $a+b+c=0$. This guarantees the conservation of "energy"

$$E = \sum_n |u_n|^2, \tag{4}$$

in the inviscid ($\nu=0$) forceless limit. As it is well known, the Sabra model has a second quadratic invariant, analogous to the helicity in fluid mechanics, of the form

$$H = \sum_n (a/c)^n |u_n|^2. \tag{5}$$

In this paper we will compare the statistics of the forced solution (with the forcing f_n restricted to the first and second shells, $n=1,2$) to the statistics of the decaying problem with $f_n = 0$ for all n. The comparison will be presented in terms of the time-independent forced structure functions S_n and time-dependent decaying structure functions F_n defined as follows:

$$\begin{aligned} &S_2(k_n) \equiv \langle |u_n|^2\rangle_f, \quad F_2(k_n,t) \equiv \langle |u_n|^2\rangle_i, \\ &S_3(k_n) \equiv \mathrm{Im}\langle u_{n-1}u_n u^*_{n+1}\rangle_f, \\ &F_3(k_n,t) \equiv \mathrm{Im}\langle u_{n-1}u_n u^*_{n+1}\rangle_i, \\ &S_4(k_n) \equiv \langle |u_n|^4\rangle_f, \quad F_4(k_n,t) \equiv \langle |u_n|^4\rangle_i, \\ &S_6(k_n) \equiv \langle |u_n|^6\rangle_f, \quad F_6(k_n,t) \equiv \langle |u_n|^6\rangle_i. \end{aligned} \tag{6}$$

Here $\langle \cdots \rangle_f$ and $\langle \cdots \rangle_i$ represent averaging with respect to realizations of the forcing and the initial conditions, respectively, for the forced and decaying problem.

The main result of the present work is that in this model there exists *strong universality*. This means that in the bulk of the inertial interval the "decaying" structure functions $F_p(k_n,t)$ take on the form

$$F_p(k_n,t) = C_p \left[\frac{\overline{\varepsilon_i}(t)}{k_0}\right]^{p/3} \lambda^{-n\,\zeta_p}, \tag{7}$$

with *the same anomalous scaling exponents* ζ_p and *the same dimensionless constants* C_p, as in the scaling laws of the "forced" structure functions

$$S_p(k_n) = C_p \left[\frac{\overline{\varepsilon_f}}{k_0}\right]^{p/3} \lambda^{-n\,\zeta_p}. \tag{8}$$

Here

$$\overline{\varepsilon_i}(t) \equiv \langle \varepsilon_n(t)\rangle_i, \quad \overline{\varepsilon_f} \equiv \langle \varepsilon_n(t)\rangle_f. \tag{9}$$

In these equations the instantaneous value of the energy flux going through nth shell (in a given realization) is

$$\varepsilon_n(t)=2k_n\{-a\lambda\,\mathrm{Im}[u_n(t)u_{n+1}(t)u^*_{n+2}] + c\,\mathrm{Im}[u_{n-1}(t)u_n(t)u^*_{n+1}]\} \qquad (10)$$

(for more details, see Ref. [27]). The only difference between Eqs. (7) and (8) is that the energy flux $\overline{\varepsilon_i}(t)$, averaged over the statistics of the initial conditions, decays in time, while the energy flux $\overline{\varepsilon_f}$, averaged over the statistics of the forcing, is time independent. Equations (7) and (8) imply that the probability distribution function (PDF) of the velocity fluctuations u_n on scales k_n within the inertial range in decaying turbulence can be obtained from the corresponding PDF in stationary forced turbulence (and vise versa). This is achieved simply by the interchange $\overline{\varepsilon_i}(t)\Leftrightarrow\overline{\varepsilon_f}$. Moreover, strong universality means that the PDF dependence on $\overline{\varepsilon_i}(t)$ for the fine scales in decaying turbulence is independent of the initial conditions, provided that the Reynolds number $\mathrm{Re}\gg 1$ and all the initial energy is concentrated in the region of large scales (first shells). Similarly, the PDF dependence on $\overline{\varepsilon_f}$ for the fine scales in forced turbulence is independent of the statistics of forcing for $\mathrm{Re}\gg 1$, provided that the forcing is concentrated in the region of large scales.

The rest of this paper is devoted to substantiating these strong propositions, which can be summarized in terms of the existence of a probability distribution function

$$\mathcal{P}(u_1,\ldots,u_N,t)=[v(t)]^{-N}P(x_1,\ldots,x_N), \qquad (11)$$

in which $x_i\equiv u_i/v(t)$ and $v(t)\propto[\epsilon_n(t)]^{1/3}$ is the corresponding velocity scale.

B. Simulations

The calculations presented bellow were carried out for the Sabra model with 28 shells, $\lambda=2$, $a=1$, $b=c=-0.5$, $k_0=2^{-4}$, and $\nu=10^{-7}$. In our simulations we employed two different types of forcing. We denote them as Forced 1 and Forced 2. Forced 1 has white noise added to the equation of the first shell. Forced 2 is forced by a Gaussian force on the first shell, which is correlated exponentially in time. In both cases the amplitude of the force in the first shell was taken to be $f_1=0.01$, while the forcing amplitude in the second shell was adjusted to reduce the helicity input $f_2=\sqrt{(-c/a)}f_1$, for more details, see Ref. [27], pp. 1813 and 1815.

In the decaying case, the total initial energy E_0 in the two first shells was kept constant. The amplitudes of the first two shell velocities were defined as $|u_1(0)|^2=\alpha E_0$ and $|u_2(0)|^2=(1-\alpha)E_0$ with α random, uniformly distributed in the interval $[0,1]$. The phases in both shells were random, uniformly distributed in the interval $[0,2\pi]$. A fourth-order Runge-Kutta scheme with adaptive time step was applied. The total energy decay was followed for nine decades in time. The statistical objects were accumulated during five decades in time.

Two recording schemes for the decaying turbulence were applied. In one case (denoted as Decay 1), the data were recorded starting after a short transient time with $E_0=10$. For this case, the data were averaged over 13 200 initial conditions. In the other case (Decay 2), the data were recorded when the energy in each realization had reached the value $E=0.1$ with $E_0=5$. These data were averaged over about 37 000 initial conditions. The decay of the total energy for two cases, plotted with an appropriate time shift, is shown in Fig. 1. It is clear that the two schemes are equivalent for the study of the advanced stages of the decay.

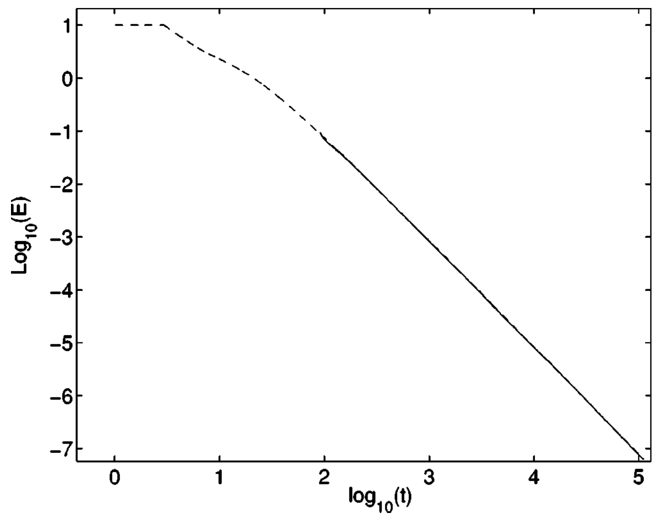

FIG. 1. The total energy decay from Decay 1 data (dashed line) and from Decay 2 data (solid line). The lines coincide within the linewidth.

C. The law of energy decay

We first discuss the total energy decay, where the total energy $E(t)$ is defined as

$$E(t)=\sum_n |u_n|^2. \qquad (12)$$

In the Navier-Stokes case the law of energy decay had been intensively studied following the influential works of Taylor [2], Kolmogorov [4,5], and Batchelor and Townsend [10,28]. For recent development, see, e.g., Ref. [8] and references therein. It was found that $E(t)\propto t^{-n}$ with the decay exponent n ranging between 1 and 5/2. This large degree of uncertainty stems from difficulties in pinpointing the energy spectrum at scales larger than the energy containing scale L. It is also not easy to determine how L depends on time. Take, for example, the case of grid turbulence in a wind tunnel. Immediately behind the grid L is of the order of the mesh size. It increases, however, with the distance from the grid. Downstream L may saturate at the wind tunnel diameter. In this regime the phenomenological analysis [8] predicts $n=2$, which is a number that is not in contradiction with experiments [11]. The same prediction ($n=2$) was reached in DNS of the Navier-Stokes equation [7] and for the GOY shell model [29]. This prediction was shown to be in agrement with numerical simulation in which L is time independent due to the special choice of initial conditions.

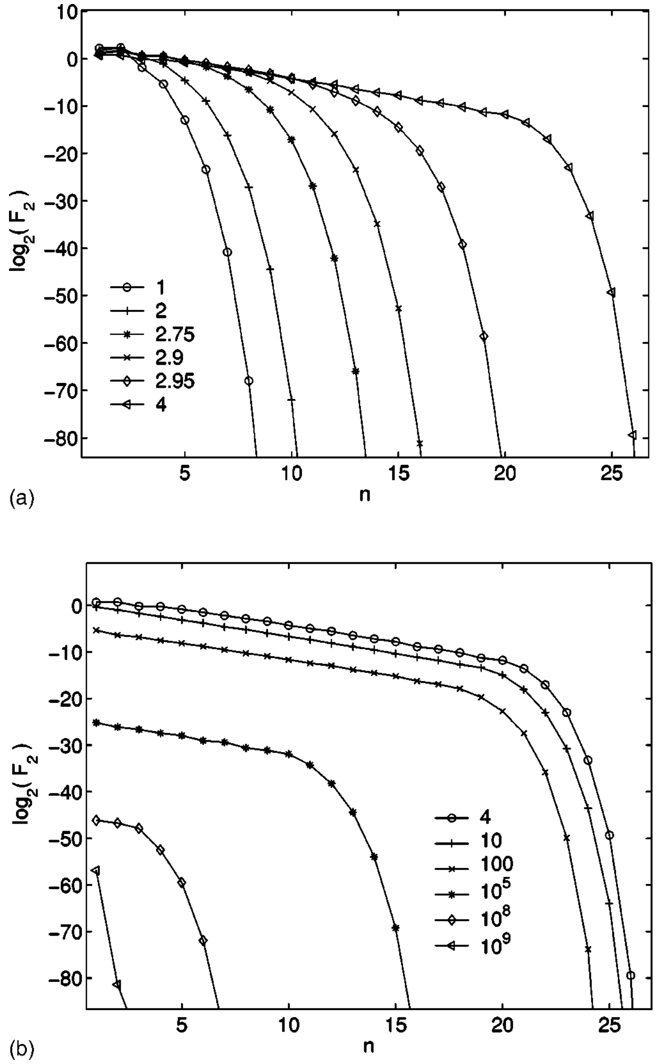

FIG. 2. The temporal behavior of F_2. The different lines correspond to the different time moments (in units of T, see the text). Upper panel: the development of the energy cascade. The energy, initially concentrated in the first two shells, cascades down to higher shell numbers with acceleration. The cascade process is completed in about $T_* \approx 3T$. For $t > T_*$ the total energy decreases. Lower panel: the decay phase. The total energy decreases for all scales and, at the same time, the viscous cutoff $k_{n,d}$ moves towards smaller shell numbers. At later stages of the decay all the energy is contained in the first shell and decays exponentially.

For the sake of completeness we review the theoretical analysis of Ref. [29], and show that our simulations of the Sabra shell model are in excellent agreement with its predictions. Consider a decaying solution with the energy initially concentrated, say, in the first two shells, see Fig. 2, upper panel. The time is measured in natural time units T that are determined by the characteristic time of the first shell, $T = 1/5\, k_0 \sqrt{E_0}$ (E_0 is the total initial energy). One sees that during one T the energy cascades down to the 6th shell, and during $2T$ down to the 12th shell. At later times the cascade process accelerates, and the energy goes from the 13th shell to "infinite" shells during a time that is roughly between $2.75\,T$ and $3\,T$. As expected, the completion of the cascade process requires a finite time T_* of a few units T. In our case $T_* \approx 3\,T$.

For $t > T_*$ the total energy of the system $E(t)$ begins to decay with

$$\frac{dE(t)}{dt} = -\overline{\varepsilon_i}(t). \tag{13}$$

Accordingly, the time-dependent Reynolds number $\mathrm{Re}(t)$, which is proportional to $\sqrt{E(t)}$, decreases, and the viscous cutoff $k_d(t)$ moves towards smaller shell numbers, as it can be seen in Fig. 2, lower panel. For example, $k_d(10^5 T) \approx 2^{12}$, $k_d(10^8 T) \approx 2^3$ and the inertial interval almost disappeares. For larger times all the energy is contained in the first shell, and it decays exponentially,

$$E(t) \propto \exp[-2\nu k_1^2 t], \tag{14}$$

following the linear part of the equation of motion for the first shell.

For intermediate times, which in our simulations span eight orders of magnitude $3\,T < t < 10^8\,T$, the slope of plots of $\log_2 F_2(k_n)$ vs n remains more or less constant. This is a manifestation of the time independence of the scaling exponents. Taking this as a fact, one immediately sees from Eq. (7) for $p=2$ that $E(t) \propto [\overline{\varepsilon_i}(t)]^{2/3}$ and hence $\overline{\varepsilon_i}(t) \propto [E(t)]^{3/2}$. Notice that this result is independent of the precise value of the scaling exponent ζ_2, anomalous or not. Thus, Eq. (13) can be presented as

$$\frac{dE(t)}{dt} = -\kappa [E(t)]^{3/2}, \tag{15}$$

with a prefactor κ, which may be expressed via the parameters of the shell model, k_0, ζ_2, and C_2. Approximately, $\kappa \simeq k_0$. In our calculations, both Decay 1 and Decay 2, $\kappa = 0.0687$, while $k_0 = 0.0625$. The solution of Eq. (15) is

$$E(t) = E_* \frac{t_*^2}{(t + t_*)^2}, \tag{16}$$

where $t_* = 2/\kappa \sqrt{E_*}$ and E_* is the integration constant. The results of the numerical simulations for the total energy, $E(t)$, (cf. Fig. 3, solid line), are in excellent agreement with Eq. (16), which is shown in the figure as a dashed line. The total energy decay was followed for nine decades in time. After 1.5 decades of transient behavior, the decay of total energy follows very closely the t^{-2} law, until at about six decades the viscous scale reaches the first shells and the decay becomes exponential in agreement with Eq. (14).

III. FORCED STRUCTURE FUNCTIONS

In this section we present results for the forced structure functions. As far as the scaling exponents are concerned, there is not much novelty in this section, the exponents are basically the same as those reported in a number of previous publications. The aspect stressed here is related to the *coefficients* C_p of the structure functions, cf. Eq. (8). We demon-

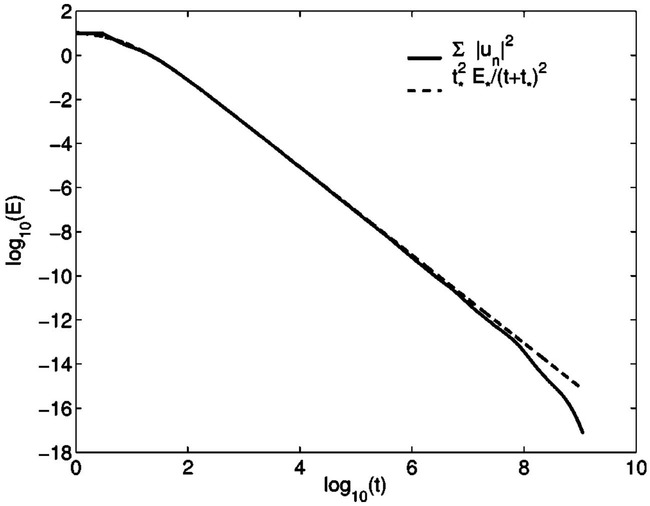

FIG. 3. The decay of the total energy $E=\Sigma_n|u_n|^2$ averaged over 3200 initial conditions. The dashed line corresponds to the decay law, Eq. (16), with $E_*=12.15$ and $t_*=8.35\,T$.

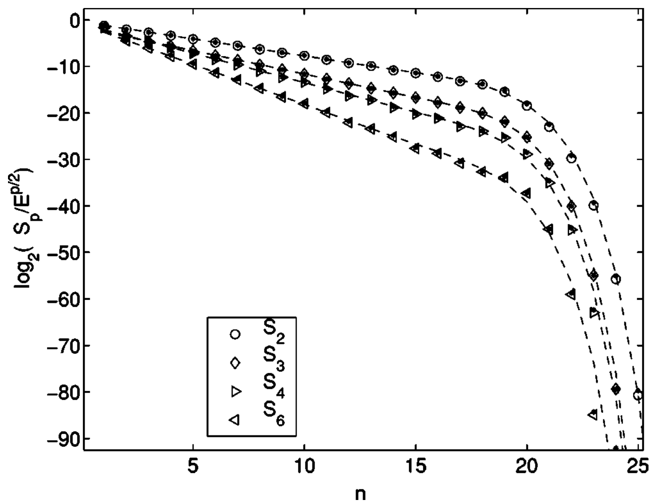

FIG. 4. The structure function for two types of the forcing. Large symbols, shown in the figure correspond to Forced 2 data. The dashed lines are the fits for the corresponding structure functions. Small black dots denote Forced 1 data. The fits for these data are not shown. Both sets were normalized by their respective total energy.

strate that these coefficients are independent of the forcing, and that the scaling form proposed in Eq. (8) is indeed universal.

In order to get an accurate determination of the scaling exponents and of the coefficients, and to be able to demonstrate strong universality, it is mandatory to fit the measured data to model functions that contain the presumed dissipative behavior. Failing to do so results in inaccuracies that may lead to confusion. As a first step in the analysis of the data, we fit the normalized structure functions $S_p/E^{p/2}$ using the fit formula Eq. (44) from Ref. [27]:

$$P_p(k_n)=\frac{A_p}{k_n^{\zeta_p}}\left(1+\alpha_p\frac{k_n}{k_{d,p}}\right)^{\mu_p}\exp\left[-\left(\frac{k_n}{k_{d,p}}\right)^x\right], \qquad (17)$$

where

$$A_p\,,\,\zeta_p\,,\,\alpha_p\,,\,\mu_p\,,\,k_{d,p}\,,\quad \text{or}\quad n_{d,p}\equiv\log_2 k_{d,p}$$

are the fit parameters and $x=\log_\lambda(1+\sqrt{5})/2$ is the exponent of the viscous range. The parameters A_p,ζ_p determine the behavior of $P_p(k_n)$ in the inertial interval, $k_{d,p}$ determines the viscous cutoff. The "auxiliary parameters" α_p,μ_p correct the behavior in the transient inertial-viscous region. The best fit is obtained by minimizing the following error function:

$$\mathcal{E}=\sqrt{\sum_n\left(1-\frac{\log_{10}P_p(k_n)}{\log_{10}S_p(k_n)}\right)^2}, \qquad (18)$$

where S_p refers to the numerically obtained data. Both sets of forced structure functions data were fitted with all the shells taken into account except the first two and the last three shells, in order to minimize boundary effects.

The quality of the fit may be seen in Fig. 4. The forced data are shown normalized by the respective total energy, but not compensated by $k_n^{\zeta_p}$, so that the different structure functions are separated and even data corresponding to Force 1 and Force 2 cases may be distinguished.

The fit procedure allows us to express the structure function in the inertial range as $S_p=A_pE^{p/2}k_n^{\zeta_p}$. To calculate the coefficients C_p we have now estimate the value of the energy flux $\overline{\varepsilon_f}$ [see Eq. (8)]. We use the exact result for S_3 to express $\overline{\varepsilon_f}$ via A_3 and the parameters of model (20). The coefficient C_p of the structure functions, other than S_3, may therefore be written as

$$C_p=\frac{A_p}{[2A_3(a-c)]^{p/3}}. \qquad (19)$$

The results are summarized in Table I, which is the central result of this paper. Before discussing them, we turn to the analysis of the decaying structure functions and add their analogous results to the table.

IV. DECAYING STRUCTURE FUNCTIONS

In this section we present results for the decaying structure functions, including the numerical support for the strong universality proclaimed in Eq. (7). We caution the reader (and whoever wants to repeat these calculations in other systems, including Navier-Stokes decaying turbulence) that the issue is fraught with subleading contributions, even in the isotropic sector. One obvious subleading term is provided by the rate of change of $F_p(k_n,t)$, which is coupled, via the infinite hierarchy of equations, to terms involving $F_{p+1}(k_m,t)$, with m of the order of n. To see this phenomenon clearly and to learn how to take it into account, we discuss first the case of the third-order structure function that can be dealt with analytically.

A. Subleading corrections to the scaling of decaying turbulence

The easiest case for theoretical analysis of the scaling behavior of the decaying structure functions $F_p(k_n,t)$ is the case $p=3$, since in the forced case $S_3(k_n)$ is known exactly [27]. For simplicity we will discuss here the helicity-free case, for which

$$S_3(k_n)=\overline{\varepsilon_f}/k_n(c-a). \tag{20}$$

We will show now that in the decaying case strong universality is realized, but only well within the inertial range. In the vicinity of the energy containing scales there are significant subleading corrections caused by the time dependence of $\overline{\varepsilon_i}(t)$.

To find these corrections, consider the equation of motion of the second-order structure function in the inertial interval (i.e., for $k_n \ll k_d$), Eq. (9) of Ref. [27]:

$$\frac{dF_2(k_n,t)}{2k_n dt}=a\lambda F_3(k_{n+1},t)+bF_3(k_n,t)+\frac{c}{\lambda}F_3(k_{n-1},t). \tag{21}$$

In the stationary case, the left-hand side of this equation vanishes and Eq. (20) is a solution. In the decaying case, let

$$F_3(k_n,t)=F_3^{(0)}(k_n,t)+\delta F_3(k_n,t), \tag{22}$$

$$F_3^{(0)}(k_n,t)=\frac{\overline{\varepsilon_i}(t)}{k_n(c-a)}.$$

In the intermediate time regime, one has that $F_2(k_n,t)\propto(t+t_*)^{-2}$, consequently

$$dF_2(k_n,t)/2k_n dt=-F_2(k_n,t)/k_n(t+t_*). \tag{23}$$

Comparing this with Eq. (21), we see that the leading solution for F_3 gains a subleading term $\delta F_3(k_n,t)\propto k_n^{-(1+\zeta_2)}$. More precisely

$$\delta F_3(k_n,t)=-\frac{F_2(k_n,t)}{(t+t_*)k_n(a\,\lambda^{-\zeta_2}+b+c\lambda^{\zeta_2})}. \tag{24}$$

Notice that $F_3^{(0)}(k_n,t)$ and $\delta F_3(k_n,t)$ have the same time dependence $(t+t_*)^{-3}$, but different scaling. As a result, their ratio is time independent; the subleading term does not become relatively smaller in time. On the other hand, it decays relatively to the leading term as k_n increases:

$$\frac{\delta F_3(k_n,t)}{F_3^{(0)}(k_n,t)}\propto\frac{1}{\lambda^{n\zeta_2}}. \tag{25}$$

While the third-order structure function is the easiest to handle, it is clear that there will always be a subleading term added to F_p from the time derivative of F_{p-1}, which appears in the infinite hierarchy of equations. Since these equations always have k_n on the right-hand side, one can immediately guess the general form of the correction to the F_p scaling, i.e.,

$$\frac{\delta F_p(k_n,t)}{F_p(k_n,t)}\propto\frac{1}{\lambda^{n(1+\zeta_{p-1}-\zeta_p)}}. \tag{26}$$

Therefore, the subleading term of the p-order structure function decreases towards small scales roughly as $\lambda^{-2n/3}$ for "normal" scaling of Kolmogorov's 1941 statistics (K41), and somewhat slower for anomalous scaling. Strong universality of the turbulent statistics is thus expected only deeply in the inertial interval.

B. Numerical results

All the calculated statistical objects were normalized by the total energy, i.e., $F_p(k_n,t)/E^{p/2}(t)$. All the normalized, compensated, decaying structure functions show a plateau. To improve the statistics, the data were first normalized by the total energy and then averaged over one-tenth of a temporal decade. For the same reasons that were explained in the case of the forced objects, the fit region for the decaying structure functions was chosen from $n=3$ to $n=n_{d,p}+5$ for the Decay 1 data and from $n=5$ to $n=n_{d,p}+5$ for the Decay 2. For $t=10^5\,T$ $n_d\approx 7$ and only a few shells may be considered as the "inertial interval." Therefore the fit parameters become unreliable. The quality of the fit can be seen in Fig. 5 for t between $20\,T$ and $10^4\,T$. For $t\leq 20\,T$ the flux equilibrium cannot be guaranteed and the coefficients C_p and the exponents may be not universal. This is definitely the case for $t\leq 10\,T$. The decaying structure functions are plotted normalized by the total energy and compensated in order to emphasis the fact that main temporal effect is the shift of $k_{d,p}$.

V. SUMMARY AND DISCUSSION

In summary, we analyzed, using the Sabra shell model of turbulence, forced structure functions $S_p(k_n)$ for two types of forcing, and decaying structure functions $F_p(k_n,t)$ for two types of initial conditions (called Decay 1 and Decay 2). For Decay 1 we considered four times, which differ in order of magnitudes ($t=20\,T$, $10^2\,T$, $10^3\,T$, and $10^4\,T$, where T is the characteristic time of the first shell). For Decay 2 we considered three different times $t=10^2\,T$, $10^3\,T$, and $10^4\,T$. In all these cases we found the scaling exponents ζ_p and the dimensionless amplitudes C_p for the three even orders $p=2$, 4, and 6. The results are collected in Table I together with our estimates of the error bars.

We can state that our results *support the conjecture of strong universality* within the numerical accuracy. Taking into account that before the data analysis presented above the raw results contained objects differing by orders of magnitude, the degree of precision of the identity of the amplitudes

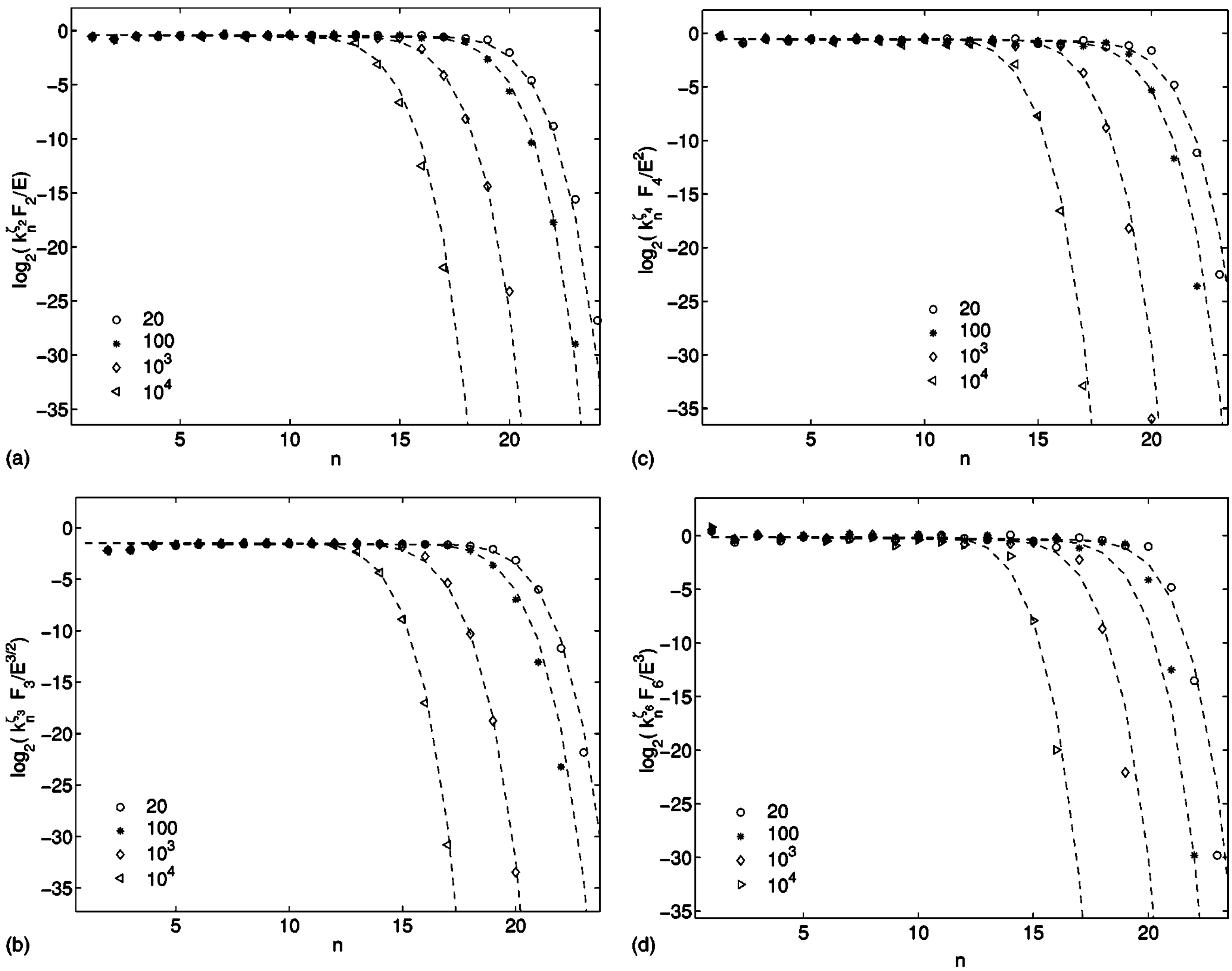

FIG. 5. The normalized compensated decaying structure functions for Decay 1 data averaged over 13 200 initial conditions and over one-tenth of the decade in time. The symbols show the calculated data at different times (defined in the legend). The dashed lines denote the fits for the corresponding structure functions. The best fit scaling exponents used for the compensation ($\zeta_2=0.728$, $\zeta_3=1$, $\zeta_4=1.254$, and $\zeta_6=1.72$) are the same as in the forced case. The Decay 2 data show similar behavior.

C_p and the exponents ζ_p shown in Table I should be taken very seriously. We propose that all the results presented by previous authors with negative indications about universality (even of the weak type) stem from problems in handling the corrections to scaling, either from anisotropy or from dissipative or other boundary effects.

It should be stressed at this point that the strong universality observed here is not expected in the much simpler problem of turbulent advection. The difference stems from the linearity of the advection problem vs the nonlinearity of the Navier-Stokes problem and its shell counterpart. In the linear advection problem one finds equations for the statistical objects that decouple for each order p. The independent p-order equations determine the anomalous scaling exponent ζ_p from solvability conditions, leaving the amplitude C_p to be found by matching the scale invariant correlation function in the inertial interval with its nonuniversal ''boundary conditions'' at the energy-containing scales. The amplitudes C_p depend, therefore, on the details of the nonuniversal forcing, and the statistics of the turbulent advection problem may exhibit weak universality only. In contrast, the nonlinearity of the Navier-Stokes equations and their shell-model counterparts leads to coupled, hierarchical equations for all the p-order statistical objects that have to be solved simultaneously. This rigid structure allows much less freedom than the linear advection problem, leading to the possibility of strong universality.

Finally, we comment on a possible theoretical support for the strong universality conjecture. We propose that a necessary condition for strong universality is the local character of the interaction, which allows to formulate (see Refs. [30–32]) the hierarchy of equations in terms of inertial-range objects only. The locality of energy transfer over scales, which is built in the shell models of turbulence, is an assumption in the Richardson-Kolmogorov cascade picture of turbulence, see, e.g., Refs. [1,4,9]. The locality of the interaction was demonstrated in Ref. [33], using the Belinicher-L'vov transformation of the Navier-Stokes equations [34], which allows to eliminate from the theory the effects of sweeping. Once we have a theory in terms of inertial-range objects, it is quite natural that amplitudes should be universal as well, up to an overall single parameter, which is the energy flux per unit time and mass. An elaboration of these ideas will be presented in a future publication. At this point we finish with the conjecture that strong universality is a property shared also by Navier-Stokes turbulence.

ACKNOWLEDGMENTS

This research was supported in part by the Israel Science Foundation administered by the Israeli Academy of Science and the European Commission under the TMR network ''Nonideal Turbulence.''

[1] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics* (MIT, Cambridge, MA, 1975).
[2] G.I. Taylor, Proc. R. Soc. London, Ser. A **151**, 421 (1935).
[3] T.D. Kármán and L. Howarth, Proc. R. Soc. London, Ser. A **164**, 192 (1938).
[4] A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 301 (1941).
[5] A.N. Kolmogorov, Dokl. Akad. Nauk SSSR **31**, 538 (1941).
[6] Y. Kaneda, in Proceedings of IUTAM Symposium on Reynolds Number Scaling in Turbulent Flow, 2002, Princeton, NJ (unpublished).
[7] V. Borue and S.A. Ozsag, Phys. Rev. E **51**, R856 (1995).
[8] D. Fukayama, T. Oyamada, T. Nakano, T. Gotoh, and K. Yamamoto, J. Phys. Soc. Jpn. **69**, 701 (2000).
[9] U. Frisch, *Turbulence: The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
[10] G.K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1953).
[11] L. Skrbek and S.R. Stalp, Phys. Fluids **12**, 1997 (2000).
[12] I. Arad, B. Dhruva, S. Kurien, V.S. L'vov, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. Lett. **81**, 5330 (1998).
[13] S. Kurien, V.S. L'vov, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. E **61**, 407 (2000).
[14] A.M. Yaglom, Atmos. Oceanic Phys. **17**, 919 (1981).
[15] B.A. Kader, Fluid Dyn. Res. **19**, 38 (1984).
[16] K.R. Sreenivasan, Phys. Fluids **7**, 2778 (1995).
[17] L. Biferale, G. Beffetta, A. Celani, A. Lanotte, F. Toschi, and M. Vergassola, e-print nlin.CD/0301040.
[18] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001), and references therein.
[19] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).
[20] I. Arad and I. Procaccia, Phys. Rev. E **63**, 056302 (2001).
[21] A. Celani and M. Vergassola, Phys. Rev. Lett. **86**, 424 (2001).
[22] I. Arad, L. Biferale, A. Celani, I. Procaccia, and M. Vergassola, Phys. Rev. Lett. **87**, 164502 (2001).
[23] Y. Cohen, T. Gilbert, and I. Procaccia, Phys. Rev. E **65**, 026314 (2002).
[24] A. Celani, T. Matsumoto, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. **88**, 054503 (2002).
[25] E.S.C. Ching, Y. Cohen, T. Gilbert, and I. Procaccia, Europhys. Lett. **60**, 369 (2002).
[26] E.S.C. Ching, Y. Cohen, T. Gilbert, and I. Procaccia, Phys. Rev. E **67**, 016304 (2003).
[27] V.S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, Phys. Rev. E **58**, 1811 (1998).
[28] G.K. Batchelor and A.A. Townsend, Proc. R. Soc. London, Ser. A **190**, 534 (1947); **193**, 539 (1948).
[29] J.-O. Hooghoudt, D. Lohse, and F. Toschi, Phys. Fluids **13**, 2013 (2001).
[30] V.S. L'vov and I. Procaccia, Physica A **257**, 165 (1998).
[31] V.I. Belinicher, V.S. L'vov, and I. Procaccia, Physica A **254**, 215 (1998).
[32] V.I. Belinicher, V.S. L'vov, A. Pomyalov, and I. Procaccia, J. Stat. Phys. **93**, 797 (1998).
[33] V.S. L'vov and I. Procaccia, Phys. Rev. E **52**, 3840 (1995).
[34] V.I. Belinicher and V.S. L'vov, Zh. Éksp. Teor. Fiz. **93**, 1269 (1987) [Sov. Phys. JETP **66**, 303 (1987)].